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Centrally extended symmetry algebra of asymptotically Gödel spacetimes

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Abstract: We define an asymptotic symmetry algebra for three-dimensional Gödel spacetimes supported by a gauge field which turns out to be the semi-direct sum of the diffeomorphisms on the circle with two loop algebras. A class of fields admitting this asymptotic symmetry algebra and leading to well-defined conserved charges is found. The covariant Poisson bracket of the conserved charges is then shown to be centrally extended to the semi-direct sum of a Virasoro algebra and two affine algebras. The subsequent analysis of three-dimensional Gödel black holes indicates that the Virasoro central charge is negative.

Keywords: Field Theories in Lower Dimensions, Black Holes, Conformal and W Symmetry.

Dedicated to Noah and Chantal

JHEP03(2007)098

Contents

1. Motivations

General relativity, although providing an elegant and very satisfactory classical description of the gravitational interaction, has left us with numbers of conceptual physical issues. One of them is related to the existence of black hole solutions, and the fact that these seem to be naturally endowed, through the laws of black hole mechanics, with a macroscopic entropy equal to the quarter of the area of their event horizon. It has therefore since then been a challenge for candidates to a quantum theory of gravity to reproduce, from first-principles, this entropy by a counting of micro-states. By now, a large number of such derivations have appeared within many distinct frameworks, e.g. in string theory or loop quantum gravity (see e.g. [1]).

Another puzzle relies on the observation that some geometries arising from Einstein's general relativity exhibit closed time-like curves. These include namely the Gödel universe [2], the Gott time-machine [3] and the region behind the inner horizon of Kerr black holes. Since the presence of closed time-like curves signals a strong breakdown of causality, Hawking advocated through his chronology protection conjecture that ultraviolet processes should prevent such geometries from forming [4]. The implications of this proposition have namely been addressed in context of string theory in a series of works (see e.g. [5 − 7], and also [8] for an extensive list of references). Also, higher-dimensional highly supersymmetric Gödel-like solutions were found in supergravity [9, 10], indicating that supersymmetry is not sufficient to discard these causally pathological solutions. Moreover, a particular issue in the dual description of gravity theories by gauge theories is the conjecture linking closed time-like curves on the gravity side and non-unitarity on the gauge side [11, 12]. It was indeed shown [11] in the context of BMPV black holes [13] that the regime of parameters in which there exists naked closed timelike curves is also the regime in which unitarity is

JHEP03(2007)098

violated in the dual CFT. Also, half BPS excitations in $AdS_5 \times S^5$ in IIB sugra can be mapped to fermions configurations [14]. Causality violation is shown to be related to Pauli exclusion principle in the dual theory [12].

It was recently shown that a new class of solutions in five-dimensional supergravity [15] could be viewed as Kerr black holes embedded in a Gödel universe. Their peculiarity lies principally in the presence of closed time-like curves in the large-radius asymptotic region so that these solutions combine the two aforementioned puzzles. Due to their unusual asymptotic behavior, being neither flat nor anti-de Sitter, the traditional methods for computing the conserved quantities associated with such black holes generically fail [16]. However, as shown in [17], their mass, angular momenta and electric charge could still consistently be defined, and were shown to satisfy both generalized Smarr formula and first law of black holes mechanics. This suggests that this class of black holes could also have an interpretation as thermodynamical objects, whose properties are worth further investigations.

A road which has revealed successful in bringing insights into properties of quantum gravity is the study of lower-dimensional systems (see e.g. [18]). A particularly eloquent example is that of the BTZ black hole, solution of 2+1 gravity with a negative cosmological constant [19, 20]. Since its discovery, it has appeared as a useful toy model to address namely the problem of black hole entropy in a simpler setting (for reviews, see e.g. [21 – 23]). In particular, Strominger's derivation of BTZ black holes' entropy exactly reproduces their geometrical Bekenstein-Hawking entropy [24]. This computation essentially relied on two earlier works: one by Brown and Henneaux [25], which showed that the canonical realization of asymptotic symmetries of AdS_3 is represented by two Virasoro algebras with non-vanishing central charge, and another by Cardy et al. via the so-called Cardy formula [26 – 28], which allows to count in the semi-classical limit the asymptotic density of states of a conformal field theory, even if the full details of the theory are not known. Application of the latter with the former central charge strikingly yields the expected number of states, even if a deep explanation of "why it works" so well is still missing so far (for a discussion on the application of the Cardy formula, see e.g. [21]).

In this note, we would like to use a similar philosophy to grasp with some properties of the aforementioned Gödel black holes through the canonical representation of their asymptotic symmetries. The theory of interest here will be (2+1)-dimensional Einstein-Maxwell-Chern-Simons theory, which can be viewed as a lower-dimensional toy-model for the bosonic part of $D = 5$ supergravity, since the field content and the couplings of both theories are similar. This theory was shown to admit as solutions Gödel universes and a three-dimensional version of Gödel black holes, displaying the same peculiar properties as their higher-dimensional counterparts [29].

Our analysis will present analogies with the one performed in AdS_3 space since there is a close relationship between AdS_3 and $3d$ Gödel space. Indeed, the latter can be seen as a squashed AdS_3 space, where the original isometry group is broken from $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{R}) \times U(1)$, as pointed out in [30]. In the context of string theory, the $3d$ Gödel metric was shown to be part of the target space of an exact two-dimensional CFT, obtained as an asymmetric marginal deformation of the $\mathrm{SL}(2,\mathbb{R})$ WZW model [6]. In this case, the

JHEP03(2007)098

effect of the deformation amounts to break the original $\widehat{\mathrm{SL}(2,\mathbb{R})} \times \widehat{\mathrm{SL}(2,\mathbb{R})}$ symmetry of the model down to $\widehat{\mathrm{SL}(2,\mathbb{R})} \times \widehat{U(1)}$. As we will show, a similar pattern will appear at the level of asymptotic symmetries.

After having briefly recalled in section 2 our general setup, we will compute in section 3 the asymptotic symmetry algebra of Gödel spaces. We then define, in section 4, a class of field configurations, which we will refer to as asymptotically Gödel space-times in three dimensions, encompassing the previously mentioned black holes solutions. In section 5, we represent the algebra of charges by covariant Poisson brackets and show that the asymptotic symmetry algebra admits central extensions. We conclude in section 6 by discussing some of the results, arguing that the sign of the Virasoro central charge is negative and that part of the entropy of the Gödel black holes can be obtained via the Cardy formula.

2. General setup

Let us start with the Einstein-Maxwell-Chern Simons theory in $2+1$ dimensions,

$$I = \frac{1}{16\pi G}\int d^3x \left[\sqrt{-g}\left(R + \frac{2}{l^2} - \frac{1}{4}F^2\right) - \frac{\alpha}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\right]. \tag{2.1}$$

The gauge parameters of the theory (ξ,λ), where ξ generates infinitesimal diffeomorphisms and λ is the parameter of $U(1)$ gauge transformations are endowed with the Lie algebra structure

$$[(\xi,\lambda),(\xi',\lambda')]_G = ([\xi,\xi'],[\lambda,\lambda']), \tag{2.2}$$

where the $[\xi,\xi']$ is the Lie bracket and $[\lambda,\lambda'] \equiv \mathcal{L}_\xi\lambda' - \mathcal{L}_{\xi'}\lambda$. We will denote for compactness the fields as $\phi^i \equiv (g_{\mu\nu}, A_\mu)$ and the gauge parameters as $f^\alpha = (\xi^\mu,\lambda)$. For a given field ϕ, the gauge parameters f satisfying

$$\mathcal{L}_\xi g_{\mu\nu} \approx 0, \qquad \mathcal{L}_\xi A_\mu + \partial_\mu\lambda \approx 0, \tag{2.3}$$

where $\approx$ is the on-shell equality, will be called the exact symmetry parameters of ϕ. Parameters $(\xi,\lambda)\approx 0$ are called trivial symmetry parameters.

In order to study the conserved charges for this theory, one canonically constructs the following 1-form in spacetime [31, 32],

$$k_{(\xi,\lambda)}[\mathrm{d}_V\phi;\phi] = k^{\mathrm{exact}}_{(\xi,\lambda)}[\mathrm{d}_V\phi;\phi] - k^s_{(\mathcal{L}_\xi g,\mathcal{L}_\xi A+d\lambda)}[\mathrm{d}_V\phi,\phi], \tag{2.4}$$

which is associated both with an arbitrary variation $\mathrm{d}_V\phi$ around ϕ and with parameters (ξ,λ). The form of k^{exact} was given in equation (65) of [29] and will not be reproduced here. The supplementary term

$$k^s_{(\mathcal{L}_\xi g,\mathcal{L}_\xi A+d\lambda)}[\mathrm{d}_V\phi,\phi] = \frac{\sqrt{-g}}{32\pi G}\left(\mathrm{d}_V g_{\mu\alpha}(D^\alpha\xi_\nu + D_\nu\xi^\alpha) - \mathrm{d}_V A_\mu(\mathcal{L}_\xi A_\nu + \partial_\nu\lambda)\right)\epsilon^{\mu\nu}{}_\alpha dx^\alpha \tag{2.5}$$

vanishes for exact symmetries but is relevant in the asymptotic context.

The form $k_{(\xi,\lambda)}[\mathrm{d}_V\phi;\phi]$ enjoys the following properties:

- For ϕ a solution of the equations of motion and $\mathrm{d}_V\phi$ a solution of the linearized equations of motion around ϕ, there is a one-to-one correspondence between the non-trivial exact symmetry parameters of ϕ and the conserved 1-forms $dk \approx 0$ of the linearized theory which are non-trivial, i.e. not exact on-shell. These forms are given by $k_{(\xi,\lambda)} = k^{\text{exact}}_{(\xi,\lambda)}$ modulo the addition of exact forms or on-shell vanishing forms. The integral of k on the circle $t = constant$, $r = constant$ then yields the finite and conserved quantity associated to (ξ, λ) that depends only on the homology class of the circle.

- The form k is constructed out of the equations of motion. It therefore does not depend on boundary terms that may be added to the lagrangian.

Additional properties of k are discussed in [33, 34]. The set of fields ϕ, $\mathrm{d}_V\phi$ and gauge parameters (ξ, λ) that satisfy the conditions

$$\oint_{S^\infty} \mathrm{d}_V k_{(\xi,\lambda)}[\mathrm{d}_V\phi, \phi] = 0, \tag{2.6}$$

$$\oint_{S^\infty} k^s_{(\mathcal{L}_\xi g, \mathcal{L}_\xi A + d\lambda)}[\mathrm{d}_V\phi, \phi] = 0, \tag{2.7}$$

where S^∞ is the circle $t = constant$, $r = constant \to \infty$ define a space of fields and parameters which we denote as the integrable space $\mathcal{I}$. In this space, we define the charges difference between the fields $\bar{\phi}$ and ϕ associated with (ξ, λ) as

$$\mathcal{Q}_{(\xi,\lambda)}[\phi, \bar{\phi}] = \oint_{S^\infty} \int_\gamma k_{(\xi,\lambda)}[\mathrm{d}_V\phi, \phi] + \mathcal{N}_{(\xi,\lambda)}[\bar{\phi}], \tag{2.8}$$

where γ is a path in field space contained in $\mathcal{I}$ and $\mathcal{N}_{(\xi,\lambda)}[\bar{\phi}]$ is an arbitrary normalization constant. Condition (2.6) ensures that the charge is independent on smooth deformations of the path γ.

Let us introduce for later convenience a subset of $\mathcal{I}$ with elements $(\phi, f[\phi])$ such that for each field ϕ the set of parameters $f[\phi]$ form a closed Lie algebra under the bracket defined in (2.2) and such that all these algebras are isomorphic. Let denote this algebra by $\mathcal{A}$. Using the conditions (2.6)-(2.7), one can then show [35] that for any solutions $\bar{\phi}$ and ϕ in the integrable space, and for any (ξ, λ), (ξ', λ') in $\mathcal{A}$, the expression

$$\mathcal{K}_{(\xi,\lambda),(\xi',\lambda')}[\bar{\phi}] = \int_{S^\infty} k_{(\xi',\lambda')}[(\mathcal{L}_\xi \bar{g}_{\mu\nu}, \mathcal{L}_\xi \bar{A}_\mu + \partial_\mu\lambda); (\bar{g}, \bar{A})] \tag{2.9}$$

is a Chevalley-Eilenberg 2-cocycle on the Lie algebra $\mathcal{A}$ and the Poisson bracket defined by

$$\left\{\mathcal{Q}_{(\xi,\lambda)}[\phi, \bar{\phi}], \mathcal{Q}_{(\xi',\lambda')}[\phi, \bar{\phi}]\right\} \equiv \oint_{S^\infty} k_{(\xi',\lambda')}[(\mathcal{L}_\xi g_{\mu\nu}, \mathcal{L}_\xi A_\mu + \partial_\mu\lambda); (g, A)] \tag{2.10}$$

obeys

$$\left\{\mathcal{Q}_{(\xi,\lambda)}[\phi, \bar{\phi}], \mathcal{Q}_{(\xi',\lambda')}[\phi, \bar{\phi}]\right\} = \mathcal{Q}_{[(\xi,\lambda),(\xi',\lambda')]_G}[\phi, \bar{\phi}] - \mathcal{N}_{[(\xi,\lambda),(\xi',\lambda')]_G}[\bar{g}, \bar{A}] + \mathcal{K}_{(\xi,\lambda),(\xi',\lambda')}. \tag{2.11}$$

JHEP03(2007)098

The central extension (2.9) is considered as trivial if it can be reabsorbed in the normalization of the charges. Saying it differently, a central charge is non-trivial if it cannot be written as a function of the bracket $[(\xi,\lambda),(\xi',\lambda')]_G$ only.

Given a solution $\bar{\phi}$, one can define an algebra $\mathcal{A}$ of asymptotic symmetries (ξ,λ) by the following conditions,

- We define the parameters (ξ,λ) such that the leading order of the expressions $\mathcal{L}_\xi \bar{g}_{\mu\nu}$ and $\mathcal{L}_\xi \bar{A}_\mu + \partial_\mu \lambda$ close to the boundary S^∞ vanishes.
- We require the expression $\mathcal{K}_{(\xi,\lambda),(\xi',\lambda')}[\bar{\phi}]$ to be a finite constant.
- We impose that the Lie bracket of two such parameters also satisfies the latter conditions.

The first condition is an adaptation of the exact symmetry equations (2.3) in the asymptotic context. In pure gravity, asymptotic Killing vectors can be defined similarly as vectors fields obeying the Killing equations to "as good an approximation as possible" as one goes to the boundary [36]. The second condition expresses finiteness and conservation of the form (2.4) integrated over S^∞ and evaluated on the background in the particular case where $\mathrm{d}_V\phi$ is $(\mathcal{L}_{\xi'}\bar{g}_{\mu\nu}, \mathcal{L}_{\xi'}\bar{A}_\mu + \partial_\mu\lambda')$. Because we will require that the phase space be left invariant under the asymptotic symmetry algebra (see section 4), such a $\mathrm{d}_V\phi$ will be tangent to the phase space. As will be required for any tangent vector to the phase space, such a $\mathrm{d}_V\phi$ has to be associated with finite and conserved charges (see also section 4). In fact, the second condition are the constraints on finiteness and conservation that one can impose already at this stage. The third condition simply ensures that the asymptotic symmetry parameters form a Lie algebra.

In what follows, we use this definition to compute the asymptotic symmetries for the Gödel spacetime and then construct a space of fields $\mathcal{F}$ consistent with $\mathcal{A}$ and satisfying the conditions (2.6)-(2.7).

3. Gödel asymptotic symmetry algebra

It was shown in [29] that the equations of motion derived from (2.1) admit the solution

$$\begin{aligned} \bar{ds}^2 &= \epsilon dt^2 - 4\alpha r dt d\varphi + \left(2r - \frac{2}{l^2}|1-\alpha^2 l^2| r^2\right) d\varphi^2 + \frac{1}{-2\epsilon r + \Upsilon^{-1} r^2} dr^2 \\ \bar{A} &= \frac{2}{l}\sqrt{|1-\alpha^2 l^2|}\, r d\varphi, \end{aligned} \tag{3.1}$$

where $\epsilon = \mathrm{sgn}(1-\alpha^2 l^2)$, $\Upsilon = \frac{l^2}{2(1+\alpha^2 l^2)}$ and $\varphi \in [0, 2\pi]$. For $\epsilon = -1$, this solution is the $3d$ part of the two parameter generalization [37] of the Gödel spacetime [2] where the stress-energy tensor of the perfect fluid supporting the metric is generated by the gauge field. For $\epsilon = +1$, the solution will be called the tachyonic Gödel spacetime because the perfect fluid supporting the metric is tachyonic. We will use this solution as background in the two sectors of the theory $\epsilon = \pm 1$.

For $\epsilon = -1$, the Gödel solution (3.1) admits 5 non-trivial exact symmetries (ξ, λ),

$$
\begin{aligned}
(\xi_{(1)}, 0) &= (\partial_t, 0),\\
(\xi_{(2)}, 0) &= (2\alpha\Upsilon\partial_t + \partial_\varphi, 0),\\
(\xi_{(3)}, 0) &= \left(\frac{2\alpha\Upsilon}{\sqrt{1+2\Upsilon/r}}\sin\varphi\partial_t - \sqrt{2\Upsilon r + r^2}\cos\varphi\partial_r + \frac{r+\Upsilon}{\sqrt{2\Upsilon r + r^2}}\sin\varphi\partial_\varphi, 0\right),\\
(\xi_{(4)}, 0) &= \left(\frac{2\alpha\Upsilon}{\sqrt{1+2\Upsilon/r}}\cos\varphi\partial_t + \sqrt{2\Upsilon r + r^2}\sin\varphi\partial_r + \frac{r+\Upsilon}{\sqrt{2\Upsilon r + r^2}}\cos\varphi\partial_\varphi, 0\right),\\
(0, \lambda_{(1)}) &= (0, 1).
\end{aligned} \tag{3.2}
$$

The four Killing vectors form a $\mathbb{R} \oplus so(2,1)$ algebra. In the case $\epsilon = +1$, only the two first vectors are Killing vectors.

Let us now compute the asymptotic symmetries of this background solution $\bar{\phi}$. They are of the form

$$
\begin{aligned}
\xi &= \chi_\xi(r)\tilde{\xi}(t,\varphi) + o(\chi_\xi(r))\\
\lambda &= \chi_\lambda(r)\tilde{\lambda}(t,\varphi) + o(\chi_\lambda(r)),
\end{aligned} \tag{3.3}
$$

for some fall-offs $\chi_\xi(r)$, $\chi_\lambda(r)$ and functions $\tilde{\xi}(t,\varphi)$, $\tilde{\lambda}(t,\varphi)$ to be determined. For such parameters, one has

$$
\mathcal{L}_\xi \bar{g}_{\mu\nu} = O(\rho_{\mu\nu}), \qquad \mathcal{L}_\xi \bar{A}_\mu + \partial_\mu\lambda = O(\rho_\mu), \tag{3.4}
$$

where $\rho_{\mu\nu}$ and ρ_μ depend on the explicit form of the parameters (3.3). Equations (3.4) are satisfied to the leading order in r when one imposes $\mathcal{L}_\xi \bar{g}_{\mu\nu} = o(\rho_{\mu\nu})$ and $\mathcal{L}_\xi \bar{A}_\mu + \partial_\mu\lambda = o(\rho_\mu)$. If one solves these equations with the highest order in r for $\chi_\xi(r)$ and $\chi_\lambda(r)$, one gets the unique solution

$$
\xi = (F(t,\varphi) + o(r^0))\partial_t + (-r\partial_\varphi\Phi(\varphi) + o(r^1))\partial_r + (\Phi(\varphi) + o(r^0))\partial_\varphi, \tag{3.5}
$$

$$
\lambda = \lambda(t,\varphi) + o(r^0), \tag{3.6}
$$

where $F(t,\varphi)$ and $\Phi(\varphi)$ are arbitrary functions. We now require the central extension (2.9) to be a finite constant. The term diverging in r in (2.9) vanishes if we impose $\xi^\varphi = \Phi(\varphi) + o(r^{-1})$. The central extension is then constant by requiring

$$
F(t,\varphi) = F(\varphi), \qquad \lambda(t,\varphi) = \lambda(\varphi). \tag{3.7}
$$

The resulting expression for (2.9) is given by

$$
K_{f,f'}[\bar{\phi}] = \frac{1}{16\pi G}\int_0^{2\pi} d\varphi \Big[2\alpha\Upsilon\partial_\varphi\Phi'\partial_\varphi^2\Phi - \frac{\epsilon}{2\alpha\Upsilon}\partial_\varphi F F' + 2\epsilon\Phi'\partial_\varphi F + \alpha\partial_\varphi\lambda\lambda' - (f \leftrightarrow f')\Big]. \tag{3.8}
$$

The asymptotic symmetries just found form a subalgebra $\mathcal{A}$ of the bracket (2.2). The asymptotic symmetries which are of the form

$$
\xi = o(r^0)\partial_t + o(r^1)\partial_r + o(r^{-1})\partial_\varphi, \qquad \lambda = o(r^0), \tag{3.9}
$$

will be considered as trivial because (i) they form an ideal of the algebra $\mathcal{A}$, (ii) for any f of the form (3.9) and $f' \in \mathcal{A}$, the associated central charge $K_{f,f'}[\bar{\phi}]$ vanishes. We define the asymptotic symmetry algebra $\mathfrak{Godel}_3$ as the quotient of $\mathcal{A}$ by the trivial asymptotic symmetries (3.9). This algebra can thus be expressed only in terms of the leading order functions $F(\varphi)$, $\Phi(\varphi)$ and $\lambda(\varphi)$. By setting $\hat{f} = [f, f']_G$, one can write the $\mathfrak{Godel}_3$ algebra explicitly as

$$\hat{F}(\varphi) = \Phi\partial_\varphi F' - \Phi'\partial_\varphi F, \qquad \hat{\Phi}(\varphi) = \Phi\partial_\varphi\Phi' - \Phi'\partial_\varphi\Phi, \qquad \hat{\lambda} = \Phi\partial_\varphi\lambda' - \Phi'\partial_\varphi\lambda. \quad (3.10)$$

A convenient basis for non-trivial asymptotic symmetries consists in the following generators

$$\begin{aligned} l_n &= \{(\xi,\lambda) \in \mathcal{A} | F(\varphi) = 2\alpha\Upsilon e^{in\varphi},\ \Phi(\varphi) = e^{in\varphi},\ \lambda(\varphi) = 0\}, \\ t_n &= \{(\xi,\lambda) \in \mathcal{A} | F(\varphi) = e^{in\varphi},\ \Phi(\varphi) = 0,\ \lambda(\varphi) = 0\}, \\ j_n &= \{(\xi,\lambda) \in \mathcal{A} | F(\varphi) = 0,\ \Phi(\varphi) = 0,\ \lambda(\varphi) = e^{in\varphi}\}. \end{aligned} \quad (3.11)$$

In terms of these generators, the $\mathfrak{Godel}_3$ algebra reads

$$\begin{aligned} i[l_m, l_n]_G &= (m-n) l_{m+n}, \\ i[l_m, t_n]_G &= -n t_{m+n}, \\ i[l_m, j_n]_G &= -n j_{m+n}, \end{aligned} \quad (3.12)$$

while the other commutators are vanishing. One can recognize the exact symmetry parameters (3.2) as a subalgebra of $\mathfrak{Godel}_3$. Indeed, one has $t_0 \sim (\xi_{(1)}, 0)$, $l_0 \sim (\xi_{(2)}, 0)$, $l_{-1} \sim (-i\xi_{(3)} + \xi_{(4)}, 0)$, $l_1 \sim (i\xi_{(3)} + \xi_{(4)}, 0)$ and $j_0 \sim (0, \lambda_{(1)})$ where $\sim$ denote the belonging to the same equivalence class of asymptotic symmetries.

In AdS_3, the exact $so(2,1) \oplus so(2,1)$ algebra is enhanced in the asymptotic context to two copies of the Witt algebra. The Gödel metric can be interpreted as a squashed AdS_3 geometry, which breaks the original $so(2,1) \oplus so(2,1)$ symmetry algebra down to $u(1) \oplus so(2,1)$ [30]. The exact Killing symmetry algebra is here enhanced to a semi-direct sum of a Witt algebra with a $\widehat{u(1)}$ loop algebra. Moreover, the gauge sector $u(1)$ is enhanced to another $\widehat{u(1)}$ loop algebra also forming an ideal of the $\mathfrak{Godel}_3$ algebra.

4. Asymptotically Gödel fields

We defined in the previous section the asymptotic symmetry algebra $\mathfrak{Godel}_3$ by a well-defined procedure starting from the background $\bar{\phi}$. One can ask which are the field configurations ϕ such that the preceding analysis leads to the same algebra (3.10) with $\bar{\phi}$ replaced by ϕ. The subset of such field configurations which is preserved under the action of the asymptotic symmetry algebra will then provide a natural definition of asymptotically

Gödel fields $\mathcal{F}$. A set of fields satisfying these conditions is given by

$$\begin{aligned}
g_{tt} &= \epsilon + r^{-1}\overset{(1)}{g}_{tt} + O(r^{-2}), \qquad g_{tr} = O(r^{-2}), \qquad g_{t\varphi} = -2\alpha r + \overset{(1)}{g}_{t\varphi} + O(r^{-1}),\\
g_{rr} &= \frac{\Upsilon}{r^2} + r^{-3}\overset{(1)}{g}_{rr} + O(r^{-4}), \qquad g_{r\varphi} = r^{-1}\overset{(1)}{g}_{r\varphi} + O(r^{-2}),\\
g_{\varphi\varphi} &= -\frac{2}{l^2}|1-\alpha^2 l^2| r^2 + r^1\overset{(1)}{g}_{\varphi\varphi} + O(r^0),\\
A_t &= -\frac{\sqrt{(1-\alpha^2 l^2)\epsilon}}{\alpha l} + r^{-1}\overset{(1)}{A}_t + O(r^{-2}),\\
A_r &= r^{-2}\overset{(1)}{A}_r + O(r^{-3}),\\
A_\varphi &= \frac{2}{l}\sqrt{|1-\alpha^2 l^2|}\, r + \overset{(1)}{A}_\varphi + O(r^{-1}),
\end{aligned} \tag{4.1}$$

where all functions $\overset{(1)}{g}_{tt}, \ldots$ depend arbitrarily on t and φ. In order for these field configurations be left invariant under the asymptotic symmetries, one has furthermore to restrict the subleading component of ξ^φ to $\xi^\varphi = \Phi(\varphi) + O(r^{-2})$. The asymptotic symmetries thus become

$$\xi = (F(\varphi) + o(r^0))\partial_t + (-r\partial_\varphi \Phi(\varphi) + o(r^1))\partial_r + (\Phi(\varphi) + O(r^{-2}))\partial_\varphi, \tag{4.2}$$

$$\lambda = \lambda(\varphi) + o(r^0), \tag{4.3}$$

and always contain the asymptotic form of the exact symmetries (3.2).

However, for the purpose of providing a well-defined representation of the asymptotic symmetry algebra, one has to restrict the definition of fields $\mathcal{F}$ by selecting those satisfying (2.6)-(2.7) and admitting finite and conserved charges. These conditions are met if the following differential equation hold,

$$\overset{(1)}{g}_{\varphi\varphi} - \epsilon\Upsilon^{-2}\overset{(1)}{g}_{rr} + 4\alpha\epsilon\overset{(1)}{g}_{t\varphi} + \frac{2(\alpha^2 l^2 - 1)}{l^2}\overset{(1)}{g}_{tt} + \frac{\epsilon}{\alpha\Upsilon}\partial_t\overset{(1)}{g}_{r\varphi} + \frac{2\epsilon\sqrt{\epsilon(1-\alpha^2 l^2)}}{\alpha l\Upsilon}(\partial_t\overset{(1)}{A}_r + \overset{(1)}{A}_t) = 0. \tag{4.4}$$

We finally define the set of asymptotically Gödel fields $\phi = (g, A)$ as those satisfying the boundary conditions (4.1) and (4.4).

In general, the asymptotic symmetries are allowed to depend arbitrarily on the fields, $(\xi[g,A], \lambda[g,A])$. They should however, by construction, obey the same algebra $\mathfrak{Godel}_3$. A basis for the asymptotic symmetries of ϕ can be written as

$$\begin{aligned}
l_n &= \{(\xi,\lambda) \in \mathcal{A} | F(\varphi) = 2\alpha\Upsilon N_l[g,A] e^{in\varphi},\ \Phi(\varphi) = e^{in\varphi},\ \lambda(\varphi) = 0\},\\
t_n &= \{(\xi,\lambda) \in \mathcal{A} | F(\varphi) = N_t[g,A] e^{in\varphi},\ \Phi(\varphi) = 0,\ \lambda(\varphi) = 0\},\\
j_n &= \{(\xi,\lambda) \in \mathcal{A} | F(\varphi) = 0,\ \Phi(\varphi) = 0,\ \lambda(\varphi) = N_j[g,A] e^{in\varphi}\},
\end{aligned} \tag{4.5}$$

The additional solution-dependent normalizations are constrained by $N_l[\bar{g},\bar{A}] = N_t[\bar{g},\bar{A}] = N_j[\bar{g},\bar{A}] = 1$ in order to match the asymptotic symmetries defined for the background.

Note that besides the background itself the asymptotically Gödel fields contain the three parameters (ν, J, Q) particle ($\epsilon = -1$) and black hole ($\epsilon = +1$) solutions found

JHEP03(2007)098

in [29],[1]

$$ds^2 = \epsilon dt^2 - 4\alpha r dt d\varphi + \left(-\frac{4GJ}{\alpha} + 8G\epsilon\nu r - \frac{2}{l^2}|1-\alpha^2 l^2| r^2\right) d\varphi^2 + \left(\frac{4GJ}{\alpha}\epsilon - 8G\nu r + \frac{r^2}{\Upsilon}\right)^{-1} dr^2$$
$$A = -\epsilon\frac{\sqrt{|1-\alpha^2 l^2|}}{\alpha l} dt + (-\frac{4G}{\alpha}Q + \frac{2}{l}\sqrt{(1-\alpha^2 l^2)\epsilon}\, r) d\varphi. \quad (4.6)$$

5. Poisson algebra

We are now ready to represent the asymptotic algebra $\mathfrak{Godel}_3$ by associated charges in the space of configurations defined in (4.1)-(4.4). An explicit computation shows that the charges associated with each generator (4.5) are in general non-vanishing. We denote these charges by $L_n \equiv \mathcal{Q}_{l_n}[\phi,\bar{\phi}]$, $T_n \equiv \mathcal{Q}_{t_n}[\phi,\bar{\phi}]$ and $J_n \equiv \mathcal{Q}_{j_n}[\phi,\bar{\phi}]$. On the contrary, all trivial asymptotic symmetries are associated with vanishing charges as it should. This provides additional justification for the quotient $\mathfrak{Godel}_3$ taken in section 3.

The central extensions (3.8) may be explicitly computed for any pair of generators of the background (3.11). The only non-vanishing terms are

$$\begin{aligned} iK_{l_m,l_n} &= \frac{c}{12} m(m^2+\epsilon)\delta_{n+m}, \\ iK_{t_m,t_n} &= \frac{\epsilon}{8G\alpha\Upsilon} m\delta_{m+n,0}. \\ iK_{j_m,j_n} &= -\frac{\alpha}{4G} m\delta_{m+n}. \end{aligned} \quad (5.1)$$

where the Virasoro-type central charge c reads

$$c = -\frac{6\alpha\Upsilon}{G} = -\frac{3\alpha l^2}{(1+\alpha^2 l^2)G}. \quad (5.2)$$

According to (2.11), the Gödel algebra is finally represented at the level of charges by the following centrally extended Poisson algebra

$$\begin{aligned} i\{L_m, L_n\} &= (m-n)(L_{m+n} - \mathcal{N}_{l_{m+n}}) + \frac{c}{12} m(m^2+\epsilon)\delta_{m+n}, \\ i\{L_m, T_n\} &= -n(T_{m+n} - \mathcal{N}_{t_{m+n}}), \\ i\{T_m, T_n\} &= \frac{\epsilon}{8G\alpha\Upsilon} m\delta_{m+n}, \\ i\{L_m, J_n\} &= -n(J_{m+n} - \mathcal{N}_{j_{m+n}}), \\ i\{J_m, J_n\} &= -\frac{\alpha}{4G} m\delta_{m+n}. \end{aligned} \quad (5.3)$$

The central extensions (5.1) are non-trivial because they cannot be absorbed into the (undetermined classically) normalizations of the generators. The L_n form a Virasoro algebra while the two loop algebras $\{t_n\}$, $\{j_n\}$ are represented by centrally extended $\widehat{u(1)}$ affine algebras.

[1] The solutions written in eq. (49)-(57) of [29] differ from the solutions written here by the change of coordinates $r^{\text{here}} = \frac{r^{\text{there}}}{\sqrt{|8G\mu^{\text{there}}|}}$, $t^{\text{there}} = \sqrt{|8G\mu^{\text{there}}|}t^{\text{there}}$, $\nu = 2\epsilon\sqrt{|8G\mu^{\text{there}}|}$.

6. Discussion

In $3d$ asymptotically anti-de Sitter spacetimes, the asymptotic symmetry algebra which consists of two copies of the Virasoro algebra [25] allows one to compute the entropy of the BTZ black hole via the Cardy formula [24]. One may wonder if an analogous derivation based on the asymptotic algebra (5.3) could be performed.

It turns out that the analysis in Gödel spacetimes is more tricky. The Gödel black holes are given in (4.6) when $\epsilon = +1$. In this case, the r coordinate has the range $-\infty < r < \infty$. The solution (4.6) displays an horizon and therefore describes a regular black hole only if the inequality

$$2G\nu^2 \geq \frac{J}{2\alpha\Upsilon} \tag{6.1}$$

holds. The tachyonic Gödel solution corresponds to $\nu = +\frac{1}{4G}$, $J = Q = 0$. Because the solutions with ν, J and Q are related by the change of coordinates $r \to -r$, $\varphi \to -\varphi$ with the solutions $-\nu$, J, $-Q$, the conserved quantity ν associated to ∂_t does not provide a satisfactory definition of mass. However, one can define the quantity $\mu = 2\epsilon G\nu^2$ which is by definition positive for black holes and which equals $-\frac{1}{8G}$ for the Gödel background. In particular, in the anti-de Sitter limit $\alpha^2 l^2 \to 1$, μ correctly reproduces the mass gap between the zero mass BTZ black hole and anti-de Sitter space. It was shown in [29] that this quantity is associated with the Killing vector $4G\epsilon\nu\partial_t$. Note also that ∂_φ is associated with $-J + \frac{Q^2}{4\alpha}$.

Choosing the normalization $N_l = 4G\nu$, the charge associated with the generator l_0 of (4.5) becomes for the black holes

$$L_0 = 2\alpha\Upsilon\mu - J + \frac{Q^2}{4\alpha} - \frac{\alpha\Upsilon}{4G} + \mathcal{Q}_{l_0}[\bar{\phi}]. \tag{6.2}$$

When $\alpha > 0$, the inequality (6.1) imposes that the spectrum of L_0 is bounded from below. The Virasoro generators L_n may then be associated with operators acting on a ground state with minimal L_0-eigenvalue. When $\alpha < 0$, one may instead consider the generators $L'_n = -L_{-n}$ satisfying also a Virasoro algebra

$$i\{L'_m, L'_n\} = (m-n)(L'_{m+n} - \mathcal{N}_{l'_{m+n}}) + \frac{c'}{12}m(m^2+1)\delta_{m+n}, \tag{6.3}$$

with $c' = -c = 6\alpha\Upsilon/G$ and for which $L'_0 = -L_0$ is also bounded from below. Remark that in any of these two cases, the classical Virasoro central charge (c for $\alpha > 0$ and c' for $\alpha < 0$) is negative, which, in whole generality, implies that the representations of this algebra are non-unitary. In the anti-de Sitter limit $\alpha^2 \to 1/l^2$, the central charge tends to minus the usual AdS_3 central charge $3l/2G$. This indicates a discontinuity in the limiting procedure.

The Bekenstein-Hawking entropy associated with the black hole solutions (4.6) is given by

$$S_{\rm BH} = 2\pi\sqrt{\alpha\Upsilon G^{-1}(2\alpha\Upsilon\mu - J)} + 2\pi\sqrt{2\alpha^2\Upsilon^2 G^{-1}\mu}. \tag{6.4}$$

Let us consider without loss of generality the case $\alpha > 0$ and define Δ_0 as the value of L_0 for the zero mass black hole $\mu = J = Q = 0$, $\Delta_0 = -\alpha\Upsilon/(4G) + \mathcal{Q}_{l_0}[\bar{\phi}]$. We observe that the first term in (6.4) may be written as $2\pi\sqrt{|c - 24\Delta_0|L_0/6}$ for $\Delta_0 = 0$ or $\Delta_0 = -\alpha\Upsilon/(2G)$ and for $Q = 0$ in the large mass $\mu \gg 1/(8G)$ limit. In the semi-classical limit $\alpha\Upsilon \gg G$, the latter formula is the Cardy formula[2] [26 − 28] for the Virasoro algebra with generators L_n.

It is possible to reproduce the second part of the entropy (6.4) via the Cardy formula by introducing operators $\hat{T}_n$ to each element of the affine algebra T_n, applying the Sugawara procedure to obtain a new Virasoro algebra $\tilde{L}_n$ with central charge $\tilde{c} = 1$ and by appropriately choosing the lowest value $\tilde{\Delta}_0$ of $\tilde{L}_0$. In this case, the effective central charge $|\tilde{c} - 24\tilde{\Delta}_0| = 6\alpha\Upsilon/G$ equals the effective central charge $|c - 24\Delta_0|$ in the initial Virasoro sector. However, this construction *a posteriori* is quite artificial.

There are several points that may deserve further investigations. It would be interesting to study the supersymmetry properties of these black holes by embedding the lagrangian (2.1) in some supergravity theory. The extension of the asymptotic symmetry algebra to a supersymmetric asymptotic symmetry algebra in the spirit of [38] would then allow one to fix the lowest value Δ_0 of L_0 undetermined classically and left ambiguous even after the matching of the entropy with the Cardy formula. Note that the naive dimensional reduction on a 2-sphere of the $5d$ minimal supergravity [9] in which Gödel black holes were studied [15] does not admit (4.6) as solutions. There are however other alternatives. Namely, it turns out that the three-dimensional Gödel black holes can be promoted to a part of an exact string theory background [39] along the lines of [40, 41], and are in particular solutions to the low energy effective action for heterotic or type II superstring theories. It could therefore be instructive to check if the present asymptotic analysis holds in this latter theories as well and then study the supersymmetry properties of these solutions.

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[2] Actually, the determination of the asymptotic density of states in a conformal field theory from the Cardy formula (see e.g. [22, 28]) seems to be meaningful only when the *effective* central charge $c_{\text{eff}} = c - 24\Delta_0$ is positive (which may encompass non unitary CFTs), which is the case for $\Delta_0 = -\alpha\Upsilon/(2G)$, and it is not obvious to us that using an absolute value is the right way to proceed when it is negative. We thank Mu-In Park and Steve Carlip for their comments on this point.

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